THE GENERALISED HAUSDORFF MEASURE OF SETS OF DIRICHLET NON-IMPROVABLE NUMBERS

PHILIP BOS, MUMTAZ HUSSAIN, AND DAVID SIMMONS

ABSTRACT. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function. A real number x is said to be ψ-Dirichlet improvable if the system

$$|qx - p| < \psi(t) \quad \text{and} \quad |q| < t$$

has a non-trivial integer solution for all large enough t. Denote the collection of such points by $D(\psi)$. Hussain-Kleinbock-Wadleigh-Wang (2018) proved the Hausdorff measure of the set $D^c(\psi)$ (the set of ψ-Dirichlet non-improvable numbers) by showing that it obeys a zero-infinity law for a large class of (essentially sub-linear) dimension functions. In this paper, we prove a new zero-infinity law valid for all dimension functions under natural non-restrictive conditions. Some of the consequences are the zero-infinity laws for all essentially sub-linear dimension functions, for some non-essentially sub-linear dimension functions, and for all dimension functions but with a growth condition on the approximating function.

1. Introduction

At its most fundamental level, the theory of Diophantine approximation is concerned with the question of how well a real number can be approximated by rationals. A qualitative answer is provided by the fact that the set of rational numbers is dense in the real numbers. Seeking a quantitative answer leads to the theory of metric Diophantine approximation. Dirichlet's theorem (1842) is the starting point in this theory.

Theorem 1.1 (Dirichlet, 1842). *Given $x \in \mathbb{R}$ and $t > 1$, there exist integers p, q such that*

$$|qx - p| \leq 1/t \quad \text{and} \quad 1 \leq q < t.$$

An easy consequence (known before Dirichlet) is the following global statement concerning the 'rate' of rational approximation to any real number.

Corollary 1.2. *For any $x \in \mathbb{R}$, there exist infinitely many integers p and $q > 0$ such that*

$$|qx - p| < 1/q.$$

A strengthening of this corollary is the classical Ψ-approximable set, here stated with a slight change of notation. Let $\Psi : [q_0, \infty) \to \mathbb{R}_+$ be a non-decreasing function with $q_0 \geq 1$ fixed. Then

$$\mathcal{K}(\Psi) := \left\{ x \in [0,1) : \left| x - \frac{p}{q} \right| < \frac{1}{q^2 \Psi(q)}, \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

A comprehensive metrical theory (Lebesgue measure, Hausdorff measure and Hausdorff dimension) for the set $\mathcal{K}(\Psi)$ is well-known, see for example [3]. To transition into the statement of results we first introduce necessary notations.

Notation. Here and throughout by a *dimension function*, we mean an increasing, continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(r) \to 0$ as $r \to 0$. For real quantities A, B that depend on parameters, we write $A \lesssim B$ if $A \leq cB$ for a constant $c > 0$ that is independent of those parameters. We write $B \asymp A$ if $A \lesssim B \lesssim A$. To simplify the presentation, we start by fixing some notation. We use $A \gg B$ to indicate that $|A/B|$ is sufficiently large. We use $\mathcal{L}(\cdot)$, $\dim_{\mathcal{H}}$ and $\mathcal{H}^f$ to denote the Lebesgue measure, Hausdorff dimension, and f-dimensional Hausdorff measure, respectively.

We state the most modern result from the seminal paper of Beresnevich-Velani [3, Theorem 2] with a slight improvement as noted in [9, Theorem 2.6].

Theorem 1.3 (Jarník, 1931). *Let ψ be a non-increasing function, and let f be a dimension function satisfying the following properties:*

$$\lim_{x\to 0} \frac{f(x)}{x} = \infty, \tag{1.1}$$

and

$$\exists\, C \geq 1 \text{ such that } \frac{f(x_2)}{x_2} \leq C\frac{f(x_1)}{x_1} \text{ whenever } x_1 < x_2 \ll 1. \tag{1.2}$$

Then

$$\mathcal{H}^f\big(\mathcal{K}(\Psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right) < \infty; \\ \infty & \text{if} \quad \sum_t tf\left(\frac{1}{t^2\Psi(t)}\right) = \infty. \end{cases}$$

Remark 1.4. *Note that the difference in the result of Beresnevich-Velani* [3, Theorem 2] *with the result* [9, Theorem 2.6] *is the weakening of the monotonicity condition "$x^{-1}f(x)$ to be decreasing" to the quasi-monotonicity condition given in* (1.2)*. The condition is only required for the divergence case and the convergence case is free from any assumptions on the dimension and the approximating functions.*

Kleinbock-Wadleigh [14] considered improvements to Dirichlet's theorem (Theorem 1.1) by considering the following set

$$D(\psi) := \left\{ x \in [0,1) : \begin{array}{l} \exists\, N \text{ such that the system } |qx-p| < \psi(t),\ |q| < t \\ \text{has a nontrivial integer solution for all } t > N \end{array} \right\}.$$

Here and throughout, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ denote a non-increasing function. For reference, a real number x will be called *ψ-Dirichlet improvable* if $x \in D(\psi)$, and elements of the complementary set, $D^c(\psi)$, will be referred to as *ψ-Dirichlet non-improvable* numbers. The main result of [14] was a Lebesgue dichotomy statement. To state their result and other results of the paper, we introduce an auxiliary function

$$\Psi(t) := \frac{t\psi(t)}{1-t\psi(t)} = \frac{1}{1-t\psi(t)} - 1. \tag{1.3}$$

Theorem 1.5 ([14], Theorem 1.8). *Let ψ be non-increasing, and suppose the function $t \mapsto t\psi(t)$ be non-decreasing and $t\psi(t) < 1$ for all large t. Then,*

$$\mathcal{L}\left(D^c(\psi)\right) = \begin{cases} 0 & \text{if} \quad \sum_t \frac{\log \Psi(t)}{t\Psi(t)} < \infty; \\ 1 & \text{if} \quad \sum_t \frac{\log \Psi(t)}{t\Psi(t)} = \infty. \end{cases}$$

The generalised f-Hausdorff measure of the set $D^c(\psi)$ has been proved in [9] for a large class of dimension functions called, the *essentially sub-linear* (ESL) dimension functions satisfying the condition,

$$\text{there exists } B > 1 \text{ such that } \limsup_{x\to 0} \frac{f(Bx)}{f(x)} < B. \tag{1.4}$$

As noted in [9, Lemma 3.1], a dimension function satisfying (1.4) also satisfies (1.1) and (1.2). This condition is clearly satisfied for the dimension functions $f(x) = x^s$ when $0 \leq s < 1$. It does not hold for $f(x) = x$ or $f(x) = x\log(1/x)$.

Theorem 1.6 ([9], 2018)**.** *Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Let f be a dimension function such that* (1.4) *is satisfied. Then*

$$\mathcal{H}^f\big(D^c(\psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right) < \infty; \\ \infty & \text{if} \quad \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right) = \infty. \end{cases}$$

Remark 1.7. *The ESL condition* (1.4) *comes into play for establishing the equivalence of the series*

$$\sum_{q=1}^{\infty} qf\Big(\frac{1}{q^2\Psi(q)}\Big) \asymp \sum_{q=1}^{\infty}\sum_{1\le p\le q} f\Big(\frac{1}{pq\Psi(q)}\Big). \tag{1.5}$$

Where the series of the right hand side arises as a covering argument for the set $D^c(\psi)$ and the series on the left hand side appears in Jarník's theorem 1.3 stated above.

Note that, even if Theorem 1.6 holds for $s = 1$, the sum condition $\sum \frac{1}{t\Psi(t)}$ of this theorem is weaker than the sum condition of Theorem 1.5, $\sum_t \frac{\log\Psi(t)}{t\Psi(t)}$, by a 'log' factor. Hence, naturally, it is desirable to establish the $\mathcal{H}^f$-measure for the set $D^c(\psi)$ for *non-essentially sub-linear* (NESL) dimension functions. A dimension function f is NESL if it satisfies,

$$\text{for every } B > 1, \limsup_{x\to 0} \frac{f(Bx)}{f(x)} \ge B. \tag{1.6}$$

The main result of this paper is the following zero-infinity dichotomy statement which is free from any ESL or NESL assumptions on the dimension functions.

Theorem 1.8. *Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Let f be a dimension function such that $x \mapsto x^{-1}f(x)$ is decreasing, and $x^{-1}f(x) \to \infty$ as $x \to 0$. Then*

$$\mathcal{H}^f\big(D^c(\psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_{k=1}^{\infty} \sum_{j<\log_2\Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right) < \infty; \\ \infty & \text{if} \quad \sum_{k=1}^{\infty} \sum_{j<\log_2\Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right) = \infty. \end{cases}$$

Some of the consequences of this theorem are the following important corollaries.

Corollary 1.9. *Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Let $a_i \in \mathbb{R}_{\ge 0}$ for $1 \le i \le n$, and*

$$f(x) = x(\log(1/x))^{a_1} x(\log\log(1/x))^{a_2} \cdots x(\log\log\cdots\log(1/x))^{a_n}. \tag{1.7}$$

Then

$$\mathcal{H}^f\big(D^c(\psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_q q\log\left(\Psi(q)\right) f\left(\frac{1}{q^2\Psi(q)}\right) < \infty; \\ \infty & \text{if} \quad \sum_q q\log\left(\Psi(q)\right) f\left(\frac{1}{q^2\Psi(q)}\right) = \infty. \end{cases}$$

Corollary 1.10. *Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Let f be a dimension function such that $x \mapsto x^{-1}f(x)$ is decreasing, and $x^{-1}f(x) \to \infty$ as $x \to 0$. Let Ψ be as in* (1.3) *such that, for all $x > 0$ and $Q > 1$, the following condition holds*

$$\Psi(Q^x) \asymp \Psi(Q), \tag{1.8}$$

where the implied constant depends only on x. Then

$$\mathcal{H}^f\big(D^c(\psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_q q\log\left(\Psi(q)\right) f\left(\frac{1}{q^2\Psi(q)}\right) < \infty; \\ \infty & \text{if} \quad \sum_q q\log\left(\Psi(q)\right) f\left(\frac{1}{q^2\Psi(q)}\right) = \infty. \end{cases}$$

Note that the condition (1.8) is trivially satisfied for approximating function such as $\Psi(Q) = \log(Q)$. The following corollary retrieves the main theorem of [9].

Corollary 1.11 ([9], Theorem 1.6)**.** *Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Let f be a dimension function such that* (1.4) *is satisfied. Then*

$$\mathcal{H}^f\big(D^c(\psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right) < \infty; \\ \infty & \text{if} \quad \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right) = \infty. \end{cases}$$

Corollary 1.12. *Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Let $a \geq 0$, and $f(x) = x\log(1/x)^a$. Then*

$$\mathcal{H}^f\big(D^c(\psi)\big) = \begin{cases} 0 & \text{if} \quad \sum_q \frac{(\log(q)^a)\log(\Psi(q))}{q\Psi(q)} < \infty; \\ \infty & \text{if} \quad \sum_q \frac{(\log(q))^a\log(\Psi(q))}{q\Psi(q)} = \infty. \end{cases}$$

From the corollaries it should be clear that Theorem 1.8 is quite powerful giving information for almost all dimension functions. However, there are certain dimension functions such as $f(x) = xe^{(\log(\frac{1}{x})^{\beta}}$ for $\beta < 1$, for which we are unable to extract conclusive information (from Theorem 1.8) for the Hausdorff measure of the set $D^c(\psi)$.

We refer the reader to [13, 15] for the metrical theory of the higher dimensional affine form version of the set $D^c(\psi)$.

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2. Continued fractions and Dirichlet improvability

The starting point for the work of Davenport & Schmidt [5] and Kleinbock & Wadleigh [14] is an observation that Dirichlet improvability is equivalent to a condition on the growth rate of partial quotients. To recall this connection, we start off with some of the basic properties of continued fractions.

Define the Gauss transformation $T : [0, 1) \to [0, 1)$ by

$$T(0) := 0, \quad T(x) := \frac{1}{x} \pmod 1, \quad \text{for } x \in (0, 1).$$

Every $x \in [0, 1)$ has a *continued fraction expansion*,

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} := [a_1(x), a_2(x), \dots]$$

where $a_1, a_2, \ldots$ are positive integers called the *partial quotients* of x and $a_n(x) = \lfloor 1/T^{n-1}(x) \rfloor$ for each $n \geq 1$. We also write $p_n/q_n = [a_1, ..., a_n]$ (p_n, q_n coprime) for the n'th *convergent* of x. With the conventions $p_{-1} = 1,\ q_{-1} = 0,\ p_0 = 0$ and $q_0 = 1$, these sequences can be generated by the following recursive relations, see [12] for further details,

$$p_{n+1} = a_{n+1}(x)p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}(x)q_n + q_{n-1}, \quad n \geq 0. \tag{2.1}$$

Thus $p_n = p_n(x), q_n = q_n(x)$ are determined by the partial quotients $a_1, \ldots, a_n$, so we may write $p_n = p_n(a_1, \ldots, a_n), q_n = q_n(a_1, \ldots, a_n)$. When it is clear which partial quotients are involved, we denote them by p_n, q_n for simplicity.

For any integer vector $(a_1, \ldots, a_n) \in \mathbb{N}^n$ with $n \geq 1$, write

$$I_n(a_1, \ldots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}$$

for the corresponding 'cylinder of order n', i.e. the set of all real numbers in $[0, 1)$ whose continued fraction expansions begin with $(a_1, \ldots, a_n)$.

We will frequently use the following well known properties of continued fraction expansions. They are explained in the standard texts [11, 12].

Proposition 2.1. *For any positive integers $a_1, \ldots, a_n$, let $p_n = p_n(a_1, \ldots, a_n)$ and $q_n = q_n(a_1, \ldots, a_n)$ be defined recursively by* (2.1). *Then:*

(P_1)

$$I_n(a_1, a_2, \ldots, a_n) = \begin{cases} \left[\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{if } \ n \text{ is even;} \\ \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right] & \text{if } \ n \text{ is odd.} \end{cases}$$

Thus, its length is given by

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2},$$

since

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n, \text{ for all } n \geq 1.$$

(P_2)

$$\frac{1}{3a_{n+1}(x)q_n^2(x)} < \left|x - \frac{p_n(x)}{q_n(x)}\right| = \frac{1}{q_n(x)(q_{n+1}(x) + T^{n+1}(x)q_n(x))} < \frac{1}{a_{n+1}q_n^2(x)}.$$

The results of [5, 14] rely crucially on the following observation,

$$\begin{aligned} x \in D(\psi) &\iff |q_{n-1}x - p_{n-1}| < \psi(q_n) \text{ for all } n \gg 1 \\ &\iff [a_{n+1}, a_{n+2}, \ldots] \cdot [a_n, a_{n-1}, \ldots, a_1] < \frac{1}{\Psi(q_n)} \text{ for all } n \gg 1. \end{aligned}$$

Where the auxiliary function Ψ is defined in (1.3). This leads to the following criterian for Dirichlet improvability.

Lemma 2.2 ([14], Lemma 2.2)). *Let $x \in [0, 1) \smallsetminus \mathbb{Q}$, and let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be non-increasing. Then*

(i) *x is ψ-Dirichlet improvable if $a_{n+1}(x)a_n(x) \leq \Psi(q_n)/4$ for all sufficiently large n.*
(ii) *x is ψ-Dirichlet non-improvable if $a_{n+1}(x)a_n(x) > \Psi(q_n)$ for infinitely many n.*

As a consequence of this lemma, and some elementary properties of continued fractions (see [9, §2], we have the inclusions

$$\mathcal{K}(3\Psi) \subseteq G_1(\Psi) \subset G(\Psi) \subset D^c(\psi) \subset G(\Psi/4),$$

where

$$\begin{aligned}
G(\Psi) &:= \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) > \Psi(q_n(x)) \text{ for i.m. } n \in \mathbb{N}\Big\},\\
G_1(\Psi) &:= \{x \in [0,1) : a_{n+1}(x) > \Psi(q_n(x)) \text{ for i.m. } n \in \mathbb{N}\},\\
\mathcal{K}(3\Psi) &:= \left\{x \in [0,1) : \left|x - \frac{p}{q}\right| < \frac{1}{3q^2\Psi(q)} \text{ for i.m. } (p,q) \in \mathbb{Z}\times\mathbb{N}\right\}.
\end{aligned}$$

Hence Theorems 1.6 and 1.8 can be restated in terms of the set $G(\Psi)$ as:

Theorem 2.3 ([9], 2018)**.** *Let $\Psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-decreasing function. Let f be a dimension function such that* (1.4) *is satisfied. Then*

$$\mathcal{H}^f\big(G(\Psi)\big) = \begin{cases} 0 & \text{if } \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right) < \infty; \\ \infty & \text{if } \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right) = \infty. \end{cases}$$

Theorem 2.4. *Let $\Psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-decreasing function. Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Let f be a dimension function such that $x \mapsto x^{-1}f(x)$ is decreasing, and $x^{-1}f(x) \to \infty$ as $x \to 0$. Then*

$$\mathcal{H}^f\big(G(\Psi)\big) = \begin{cases} 0 & \text{if } \sum_{k=1}^{\infty} \sum_{j<\log_2\Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right) < \infty; \\ \infty & \text{if } \sum_{k=1}^{\infty} \sum_{j<\log_2\Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right) = \infty. \end{cases}$$

It can readily be seen that the divergence case of Jarnik's theorem played a pivotal role in proving the divergence case of the Theorem 1.6. Further, it is worth pointing out that the difference set $G(\Psi) \setminus \mathcal{K}(3\Psi)$ is non-trivial as the Hausdorff dimension of this set is the same as that of $G(\psi)$ as proved in [1]. We refer the reader to [2, 8] for the Lebesgue measure and Hausdorff dimension results for a generalised form of the set $G(\Psi)$.

For any $\tau \geq 0$, as a consequence of Theorem 2.3, the Hausdorff dimension of the sets of Dirichlet non-improvable numbers with prescribed order of approximation τ can be deduced. Define the set with *order* τ as

$$G(\tau) := \left\{x \in [0,1) : \limsup_{n\to\infty} \frac{\log\left(a_n(x)a_{n+1}(x)\right)}{\log q_n(x)} \geq \tau\right\}$$

and *exact order* τ as

$$G(\tau_{\text{exac}}) := \left\{x \in [0,1) : \limsup_{n\to\infty} \frac{\log\left(a_n(x)a_{n+1}(x)\right)}{\log q_n(x)} = \tau\right\}.$$

Corollary 2.5.

$$\dim_{\mathcal{H}} G(\tau) = G(\tau_{\text{exac}}) = \frac{2}{\tau+2}.$$

We refer the reader to [7] for the Hausdorff dimension of level sets obtained by replacing lim sup with lim in the definition of $G(\tau_{\text{exac}})$.

Finally, for completeness we give a very brief introduction to Hausdorff measures and dimension. For further details we refer to the beautiful texts [4, 6].

2.1. **Hausdorff measure and dimension.**

Let f be a dimension function and let $E \subset \mathbb{R}^n$. Then, for any $\rho > 0$ a countable collection $\{B_i\}$ of balls in $\mathbb{R}^n$ with diameters $\mathrm{diam}(B_i) \le \rho$ such that $E \subset \bigcup_i B_i$ is called a ρ-cover of E. Let

$$\mathcal{H}^f_\rho(E) = \inf \sum_i f\big(\mathrm{diam}(B_i)\big),$$

where the infimum is taken over all possible ρ-covers $\{B_i\}$ of E. It is easy to see that $\mathcal{H}^f_\rho(E)$ increases as ρ decreases and so approaches a limit as $\rho \to 0$. This limit could be zero or infinity, or take a finite positive value. Accordingly, the *Hausdorff s-measure* $\mathcal{H}^f$ of E is defined to be

$$\mathcal{H}^f(E) = \lim_{\rho \to 0} \mathcal{H}^f_\rho(E).$$

It is easily verified that Hausdorff measure is monotonic and countably sub-additive, and that $\mathcal{H}^f(\varnothing) = 0$. Thus it is an outer measure on $\mathbb{R}^n$.

In the case when $f(x) = x^s$ for some $s \ge 0$, we write $\mathcal{H}^s$ for $\mathcal{H}^f$. Furthermore, for any subset E one can verify that there exists a unique critical value of s at which $\mathcal{H}^s(E)$ 'jumps' from infinity to zero. The value taken by s at this discontinuity is referred to as the *Hausdorff dimension of* E and is denoted by $\dim_{\mathcal{H}} E$; i.e.,

$$\dim_{\mathcal{H}} E := \inf\{s \ge 0 : \ \mathcal{H}^s(E) = 0\}.$$

When $s = n$, $\mathcal{H}^n$ coincides with standard Lebesgue measure on $\mathbb{R}^n$.

Computing Hausdorff dimension of a set is typically accomplished in two steps: obtaining the upper and lower bounds separately.

Upper bounds often can be handled by finding appropriate coverings. When dealing with a limsup set, one usually applies the Hausdorff measure version of the famous Borel-Cantelli lemma (see Lemma 3.10 of [4]).

Proposition 2.6. *Let $\{B_i\}_{i \ge 1}$ be a sequence of measurable sets in $\mathbb{R}^n$ and suppose that for some dimension function f, $\sum_i f\big(\mathrm{diam}(B_i)\big) < \infty$. Then $\mathcal{H}^f(\limsup_{i\to\infty} B_i) = 0$.*

3. Proof of Theorem 2.4

3.1. **The convergence case.** We are given that the series

$$\sum_{k=1}^{\infty} \sum_{j < \log_2 \Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right) \tag{3.1}$$

converges. We can assume that $\Psi(t) \ge 1$ for all $t \gg 1$. Otherwise, $\Psi(t) < 1$ for all large t since we have assumed Ψ to be non-decreasing. Then it is obvious that the set

$$G_1(\Psi) = \{x \in [0,1) : a_{n+1}(x) \ > \ \Psi(q_n) \ \text{ for i.m. } n \in \mathbb{N}\}$$

and thus

$$G(\Psi) = \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \ge \Psi(q_n) \ \text{ for i.m. } n \in \mathbb{N}\Big\},$$

contains all irrational numbers in $[0,1]$, and that the sum (3.1) diverges. Since Ψ is increasing and from (2.1) it follows that $q_n \geq a_n q_{n-1}$. We notice some obvious inclusions,

$$\begin{aligned} G(\Psi) =& \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \text{ for i.m. } n \in \mathbb{N}\Big\} \\ \subseteq& \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(a_n q_{n-1}) \text{ for i.m. } n \in \mathbb{N}\Big\} \\ \subseteq& \bigcup_{n=N}^{\infty} \bigcup_{a_1,\dots,a_n} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\dots,a_n,a_{n+1}) \\ =& \mathcal{A}_1(\Psi) \cup \mathcal{A}_2(\Psi). \end{aligned}$$

Where

$$\mathcal{A}_1(\Psi) = \bigcup_{n=N}^{\infty} \bigcup_{a_1,\dots,a_n} \bigcup_{a_n \leq \Psi(q_{n-1})} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\dots,a_n,a_{n+1}),$$

$$\mathcal{A}_2(\Psi) = \bigcup_{n=N}^{\infty} \bigcup_{a_1,\dots,a_n} \bigcup_{a_n > \Psi(q_{n-1})} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\dots,a_n,a_{n+1}).$$

3.1.1. *Covering for $\mathcal{A}_1(\Psi)$.* To estimate the Hausdorff measure of the set $\mathcal{A}_1(\Psi)$, we proceed as follows. Let

$$J_n(a_1,\dots,a_n) := \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\dots,a_n,a_{n+1}).$$

Using (P_1) in Proposition 2.1 and the recursive relation (2.1), the diameter of $J_n(a_1,\dots,a_n)$ can be bounded as follows:

$$\begin{aligned} |J_n(a_1,\dots,a_n)| =& \sum_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} \left| \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} - \frac{(a_{n+1}+1)p_n + p_{n-1}}{(a_{n+1}+1)q_n + q_{n-1}} \right| \\ \leq& \left| \frac{\frac{\Psi(a_n q_{n-1})}{a_n} p_n + p_{n-1}}{\frac{\Psi(q_n)}{a_n} q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{1}{\left(\frac{\Psi(a_n q_{n-1})}{a_n} q_n + q_{n-1}\right) q_n} \\ \leq& \frac{1}{\Psi(a_n q_{n-1}) a_n q_{n-1}^2}. \end{aligned}$$

Let $Q > 1$ and $Q < q_{n-1} \leq 2Q$. Then we can bound the diameter of $J_n(a_1,\dots,a_n)$ as

$$|J_n(a_1,\dots,a_n)| \ll \frac{1}{\Psi(a_n Q) a_n Q^2}.$$

Hence, the cost of the cover when $a_n < \Psi(q_{n-1})$, is

$$\sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right).$$

In the case $a_n > \Psi(q_{n-1})$, the cost of the cover is given by

$$f\left(\frac{1}{Q^2\Psi(Q)}\right).$$

Since $Q > 1$, it follows from equation ((P_1)) that for each window $[Q, 2Q]$, there are at most Q^2 cylinders I_n of length comparable (up to a constant) to Q^{-2}. Multiplying the cost of the cover given above by Q^2 which are the number of such intervals, and then summing over all the windows $Q = 2^k$, we have

$$\sum_{Q=2^k;k\geq 1} Q^2 \sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right) + \sum_{Q=2^k;k\geq 1} Q^2 f\left(\frac{1}{Q^2\Psi(Q)}\right).$$

Applying Cauchy's condensation test on the second term, and rewriting the first term gives the total cost as

$$\sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2 A\Psi(QA)}\right) + \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right).$$

The second term is clearly smaller than the first term so we can ignore it. Thus, using Proposition 2.6, we have

$$\mathcal{H}^f(\mathcal{A}_1(\Psi)) = 0 \quad \text{if} \quad \sum_{k=1}^{\infty} \sum_{j<\log_2 \Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right) < \infty.$$

3.1.2. *Covering for $\mathcal{A}_2(\Psi)$.* Next we notice that the set $\mathcal{A}_2(\Psi)$ is a subset of the Jarník set, that is, it is contained in the set $G_1(\Psi)$. To see this note that if $a_n \geq \Psi(q_{n-1})$ then it follows that $a_{n+1} \geq \Psi(q_{n-1})$ for infinitely many n. This in turn implies that for any dimension function f

$$\mathcal{H}^f(\mathcal{A}_2(\Psi)) \leq \mathcal{H}^f(G_1(\Psi)) \leq \mathcal{H}^f(\mathcal{K}(\Psi)) = 0 \iff \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right) < \infty.$$

However notice that the series above is smaller than the one we are claiming in our theorem. Hence combining both the coverings for $\mathcal{A}_1(\Psi)$ and $\mathcal{A}_2(\Psi)$, and by using Proposition 2.6, we conclude that

$$\mathcal{H}^f(G(\Psi)) = 0 \quad \text{if} \quad \sum_{k=1}^{\infty} \sum_{j<\log_2 \Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right) < \infty.$$

3.2. **The divergence case.** For the divergence case, we appeal to a recent criterion introduced by Hussain-Simmons [10]. We briefly state this criterion below and then use it to prove the divergence case in the subsequent subsection.

3.2.1. *A generalised Hausdorff measure criterion.* Let X be a metric space. For $\delta > 0$, a measure μ is *Ahlfors δ-regular* if and only if there exist positive constants $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that the inequality

$$c_1 r^\delta \leq \mu(B(x,r)) \leq c_2 r^\delta$$

holds for every ball $B := B(x,r)$ in X of radius $r \leq r_0$ centred at $x \in \mathrm{Supp}(\mu)$, where $\mathrm{Supp}(\mu)$ denotes the topological support of μ. The space X is called Ahlfors δ-regular if there is an Ahlfors δ-regular measure whose support is equal to X. If X is Ahlfors δ-regular, then so is the $\mathcal{H}^\delta$ measure restricted to X i.e. $\mathcal{H}^\delta \upharpoonright X$.

Theorem 3.1 (Hussain-Simmons, [10])**.** *Fix $\delta > 0$, let $(B_i)_i$ be a sequence of open sets in an Ahlfors δ-regular metric space X, and let f be a dimension function such that $r \mapsto r^{-\delta} f(r)$ is decreasing, and $r^{-\delta} f(r) \to \infty$ as $r \to 0$. Fix $C > 0$, and suppose that the following hypothesis holds:*

(*) *For every ball* $B_0 \subset X$ *and for every* $N \in \mathbb{N}$*, there exists a probability measure* $\mu = \mu(B_0, N)$ *with* $\mathrm{Supp}(\mu) \subset \bigcup_{i \geq N} B_i \cap B_0$*, such that for every ball* $B = B(x, \rho) \subset X$*, we have*

$$\mu(B) \lesssim \max\left(\left(\frac{\rho}{\operatorname{diam} B_0}\right)^\delta, \frac{f(\rho)}{C}\right). \tag{3.2}$$

Then for every ball B_0*,*

$$\mathcal{H}^f\left(B_0 \cap \limsup_{i\to\infty} B_i\right) \gtrsim C.$$

In particular, if the hypothesis (*) holds for all C*, then*

$$\mathcal{H}^f\left(B_0 \cap \limsup_{i\to\infty} B_i\right) = \infty.$$

The condition "$r^{-\delta} f(r) \to \infty$ as $r \to 0$" is a natural condition which implies that $\mathcal{H}^f(B) = \infty$. The hypothesis (*) is the main ingredient of this theorem and, roughly speaking, this gives a systematic way of constructing the probability measure on the limsup set.

3.2.2. *Proof of the divergence case.* Now we are in a position to proof the divergence case. We will prove, in particular, the following generalised form from which the divergence case of Theorem 1.8 readily follows.

A collection of maps $u = (u_a)_{a\in E}$ is a Gauss Iterated Function System (GIFS) on $\mathbb{R}$, if:

- E is a countable (finite or infinite) index set, which is referred to as an alphabet;
- $X \subseteq \mathbb{R}$ is a nonempty compact set which is equal to the closure of its interior;
- for all $a \in E, u_a(X) \subset X$.

Theorem 3.2. *Let* $(u_a)_{a\in E}$ *be the Gauss iterated function system. For each finite word* $\omega \in E^*$ *and* $a \leq \Psi(Q_\omega)$ *let*

$$S_{\omega a} = u_{\omega a}([0, a/\Psi(Q_\omega a)]).$$

Let f *be a dimension function such that* $\sum_{\omega,a} f(\operatorname{diam} S_{\omega a})$ *diverges. Then*

$$\mathcal{H}^f\left(\limsup_{\omega,a} S_{\omega a}\right) = \infty.$$

To apply theorem 3.2 to derive the Hausdorff measure of the set $G(\Psi)$, notice that the set $S_{\omega a}$ corresponds to the set

$$\bigcup_{a_n \leq \Psi(q_{n-1})} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}).$$

Hence

$$\limsup_{\omega,a} S_{\omega a} \subseteq G(\Psi).$$

As in the convergence case for the set $\mathcal{A}_1(\Psi)$, let $Q > 1$ and $Q < q_{n-1} \leq 2Q$. Then we have

$$\infty = \sum_{\omega,a} f(\operatorname{diam} S_{\omega a}) \asymp \sum_{Q=2^k; k\geq 1} Q^2 \sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right) \asymp \sum_{k=1}^{\infty} \sum_{j<\log_2 \Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right).$$

Proof of Theorem 3.2. Fix $B_0 \subset [0,1]$ and $N \in \mathbb{N}$, and we will construct the measure $\mu = \mu(B_0, N)$ such that the hypothesis (*) in Theorem 3.1 holds.

For each $k, \ell \in \mathbb{N}$ let

$$A_{k,\ell} = \{(\omega, a) : 2^k \le Q_\omega < 2^{k+1},\ \ 2^\ell \le a < 2^{\ell+1}\}.$$

Then for all $(\omega, a) \in A_{k,\ell}$, we have $\operatorname{diam}(S_{\omega a}) \asymp \rho_{k,\ell}$. Thus

$$\sum_{k,\ell} \#(A_{k,\ell}) f(\rho_{k,\ell}) = \infty.$$

Let

$$A'_{k,\ell} = \{(\omega, a) \in A_{k,\ell} : S_{\omega a} \subset B_0\}.$$

Claim 3.3.

$$\#(A'_{k,\ell}) \gtrsim \#(A_{k,\ell})|B_0|$$

for all $k, \ell \ge N_0$, for some sufficiently large N_0.

Proof. Indeed, consider the set

$$\left\{\tau b : Q_\tau < 2^k \le Q_{\tau b}\right\}.$$

Fix τ such that $Q_\tau < 2^k$. Since $Q_{\tau b} = bQ_\tau + Q_{\tau'}$, where τ' is τ minus its last symbol, it follows that there are approximately $2^k/Q_\tau$ values of b such that $2^k \le Q_{\tau b} < 2^{k+1}$. On the other hand, the set

$$\bigcup_{b: 2^k \le Q_{\tau b}} u_{\tau b}([0,1]) \tag{3.3}$$

has measure approximately $(2^k/Q_\tau)^{-1} Q_\tau^{-2} = 2^{-k}/Q_\tau$. Since the sets (3.3) form a disjoint cover of $[0, 1]$, it follows that

$$\begin{aligned}
\#(A'_{k,\ell}) &\ge 2^\ell \#\{\omega : 2^k \le Q_\omega < 2^{k+1},\ S_\omega \cap B_0 \ne \emptyset\} \\
&\ge 2^\ell \sum_{\substack{\tau \\ Q_\tau < 2^k \\ S_\tau \cap B_0 \ne \emptyset}} \left| \bigcup_{b: 2^k \le Q_{\tau b}} u_{\tau b}([0,1]) \right| \\
&\asymp 2^\ell |B_0| \# \left\{\tau : Q_\tau < 2^k,\ S_\tau \cap B_0 \ne \emptyset\right\} \\
&\asymp \#(A_{k,\ell})|B_0|.
\end{aligned}$$

□

It follows that

$$\sum_{k,\ell \ge N_0} \#(A'_{k,\ell}) f(\rho_{k,\ell}) = \infty.$$

Fix N_1 such that

$$\Omega = \sum_{N_0 \le k,\ell \le N_1} \#(A'_{k,\ell}) f(\rho_{k,\ell}) \ge C$$

and define the measure μ as follows:

$$\mu = \frac{1}{\Omega} \sum_{N_0 \le k,\ell \le N_1} \sum_{(\omega,a) \in A'_{k,\ell}} f(\rho_{k,\ell}) \lambda_{S_{\omega a}},$$

where λ_A is normalized Lebesgue measure on a set A.

Let $B = u_\tau([1/b_1, 1/b_2])$ for some τ, b_1, b_2 (possibly $b_2 = 1$). Next we estimate $\mu(B)$ and show that it satisfies (3.2). Let

$$A''_{k,\ell} = \{(\omega, a) \in A_{k,\ell} : S_{\omega a} \subset B\}.$$

Then clearly

$$\#(A''_{k,\ell}) \lesssim \#(A_{k,\ell})|B| \asymp \#(A'_{k,\ell})\frac{|B|}{|B_0|}$$

and thus

$$\frac{1}{\Omega}\sum_{N_0\leq k,\ell\leq N_1}\sum_{(\omega,a)\in A''_{k,\ell}} f(\rho_{k,\ell})\lambda_{S_{\omega a}}(B) \lesssim \frac{1}{\Omega}\sum_{N_0\leq k,\ell\leq N_1}\#(A'_{k,\ell})\frac{|B|}{|B_0|}f(\rho_{k,\ell}) = \frac{|B|}{|B_0|}.$$

Now for all (ω, a) such that $S_{\omega a}\cap B \neq \emptyset$, we have either $S_{\omega a}\subset B$ or $B\subset S_{\omega a}$. If the latter case never holds, then we are done. Otherwise, we have

$$\mu(B) = \frac{1}{\Omega}f(\rho_{k,\ell})\lambda_{S_{\omega a}}(B),$$

where $(\omega,a)\in A'_{k,\ell}$ is chosen so that $B\subset S_{\omega a}$. Since $\Omega \geq C$ and $\rho_{k,\ell}\asymp |S_{\omega a}|$, we have

$$\mu(B)\lesssim \frac{f(\operatorname{diam} B)}{C}$$

in this case.

Now let B_1 be an arbitrary ball, and let ω be the longest word such that $B\subset S_\omega$. If there exist distinct $n_1,n_2\in\mathbb{N}$ such that $u_\omega(1/n_i)\in B_1$, then let $b_1 = \lfloor 1/\max(B_1)\rfloor$ and $b_2 = \lceil 1/\min(B_1)\rceil$ and let B be as above. Then $\operatorname{diam}(B_1)\asymp \operatorname{diam}(B)$ and $B_1\subset B$, so it follows from the previous paragraph that

$$\mu(B_1)\lesssim \frac{f(\operatorname{diam} B_1)}{C}.$$

On the other hand, if there do not exist such distinct n_1, n_2, then by the maximality of ω there exists one such $n\in\mathbb{N}$ such that $u_\omega(1/n)\in B_1$. Let $n_1 = n$ and $n_2 = n+1$, and let b_i be maximal such that

$$B_1 \subset u_{n_1}([0,1/b_1])\cup u_{n_2}([0,1/b_2]).$$

The argument of the previous paragraph shows that for all $i = 1,2$,

$$\mu(u_{\omega n_i}([0,1/b_i])) \lesssim \frac{f(\operatorname{diam} u_{\omega n_i}([0,1/b_i]))}{C}$$

and on the other hand, the maximality of b_i gives

$$\operatorname{diam}(u_{\omega n_i}([0,1/b_i]))\asymp \operatorname{diam}(B_1).$$

It follows that

$$\mu(B_1)\lesssim \frac{f(\operatorname{diam} B_1)}{C}.$$

Hence the proof of Theorem 3.2 is complete. □

4. Proofs of the corollaries

4.1. Proof of Corollary 1.9. Recall that

$$\sum_{\substack{k\geq 1\\ Q=2^k}}\sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2A\Psi(QA)}\right) \asymp \sum_{j<\log_2\Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right).$$

The proof of the corollary follows if we show that, for the dimension function f satisfying (1.7), the following two series are equivalent

$$\sum_{\substack{k\geq 1\\ Q=2^k}}\sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2A\Psi(QA)}\right) \asymp \sum_q q\log(\Psi(q))\, f\left(\frac{1}{q^2\Psi(q)}\right). \tag{4.1}$$

Since $x^{-1}f(x)$ is monotonic, we have that

$$Af\left(\frac{x}{A}\right) \leq A^2 f\left(\frac{x}{A^2}\right).$$

Using this inequality we have

$$\begin{aligned}
\sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2 A\Psi(QA)}\right) &\ll \sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A^2 f\left(\frac{1}{Q^2 A^2\Psi(QA)}\right)\\
&\asymp \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(R)}} R^2 f\left(\frac{1}{R^2\Psi(R)}\right) \text{ by setting } R = QA\\
&\asymp \sum_{q} q\log\left(\Psi(q)\right) f\left(\frac{1}{q^2\Psi(q)}\right).
\end{aligned}$$

For the reverse inequality, let f satisfy the condition (1.7). For convenience, let $f(x) = x(\log(1/x))^{a_1}$ with $a_1 \geq 0$. Clearly the series are equivalence for $a_1 = 0$, therefore we assume that $a_1 > 0$. For this, we show that $f(x) \gg Af(x/A)$, where $x = \frac{1}{Q^2A^2\Psi(QA)}$.

Clearly $Ax \leq 1$.

$$\begin{aligned}
&\Longrightarrow \log(A/x) \leq \log(1/x)^2\\
&\Longrightarrow \log(A/x)^{a_1} \leq 2^{a_1}\log(1/x)^{a_1}.
\end{aligned}$$

Thus,

$$x(\log(1/x))^{a_1} \gg A\frac{x}{A}(\log(A/x))^{a_1}.$$

Hence,

$$\sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2 A\Psi(QA)}\right) \gg \sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A^2 f\left(\frac{1}{Q^2 A^2\Psi(QA)}\right).$$

Remark 4.1. *We suspect that the claim* (4.1) *should be true for any NESL dimension functions and for that the full potential of the NESL condition* (1.6) *should be exploited. That is, f is NESL if and only if for any $B > 1$ there exists a sequence $x_n \to 0$ such that $f(Bx_n)/f(x_n) \geq B$. Since $x^{-1}f(x)$ is decreasing, for any $B \geq 1$, there exists a sequence $x_n \to 0$ such that $\frac{f(Bx_n)}{Bf(x_n)} \to 1$. The main difficulty is that the sequence can be very sparse and there is no concrete information on the behaviour of f for such sequences. We refer the reader to the first version of this article on the arXiv (https://arxiv.org/abs/2010.14760, pp. 8-9) for an alternative argument that may be useful.*

4.2. **Proof of Corollary 1.10.** Just like the proof of the Corollary 1.9, we show the equivalence of the series (4.1) for the approximating function Ψ assuming a growth condition (1.8), i.e. for all $x > 0$ and $Q > 1$, $\Psi(Q^x) \asymp \Psi(Q)$. In this case, we have

$$\begin{aligned}
\sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2 A\Psi(QA)}\right) &\asymp \sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2 A\Psi(QA^{1/2})}\right)\\
&\asymp \sum_{\substack{k\geq 1\\ R=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(R)}} R^2 f\left(\frac{1}{R^2\Psi(R)}\right) \text{ by setting } R = QA^{1/2}\\
&\asymp \sum_{q} q\log\left(\Psi(q)\right) f\left(\frac{1}{q^2\Psi(q)}\right)
\end{aligned}$$

4.3. **Proof of Corollary 1.11.** Let f be an ESL dimension function. We show that

$$\sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2 A\Psi(QA)}\right) \asymp \sum_q q f\left(\frac{1}{q^2\Psi(q)}\right) \asymp \sum_q q\log\left(\Psi(q)\right) f\left(\frac{1}{q^2\Psi(q)}\right).$$

Recall the covering argument for $\mathcal{A}_1(\Psi)$

$$|J_n(a_1,\ldots,a_n)| \leq \frac{1}{\Psi(a_n q_{n-1})a_n q_{n-1}^2} \ll \frac{1}{\Psi(q_n)a_n q_{n-1}^2} \asymp \frac{1}{\Psi(q_n)q_{n-1}q_n}.$$

Next following the same method as in [9, pp. 512-13], we arrive at the

$$\mathcal{H}^f\big(\mathcal{A}_1(\Psi)\big) \leq 2\liminf_{N\to\infty} \sum_{q\geq 2^{(N-1)/2}} \sum_{\frac{q}{\Psi(q)}<p<q} f\left(\frac{1}{pq\Psi(q)}\right).$$

Following [9], the restriction on p follows from the fact that, since $a_n < \Psi(q_{n-1})$,

$$q_n \leq (a_n+1)q_{n-1} \ll \Psi(q_{n-1})q_{n-1} < \Psi(q_n)q_{n-1} \Longrightarrow q_{n-1} > \frac{q_n}{\Psi(q_n)}.$$

It can be seen that if

$$\sum_{q=1}^{\infty} \sum_{\frac{q}{\Psi(q)}<p\leq q} f\left(\frac{1}{pq\Psi(q)}\right) < \infty,$$

then it readily follows from Proposition 2.6 that $\mathcal{H}^f\big(\mathcal{A}_1(\Psi)\big) = 0$. Next we compare all these series by using the condition (1.4),

$$\begin{aligned}
\sum_{q=1}^{\infty} q\log(\Psi(q)) f\left(\frac{1}{q^2\Psi(q)}\right) &= \sum_{q=1}^{\infty} \sum_{\frac{q}{\Psi(q)}<p\leq q} \frac{q}{p} f\left(\frac{1}{q^2\Psi(q)}\right)\\
&\overset{(1.4)}{\geq} \sum_{q=1}^{\infty} \sum_{\frac{q}{\Psi(q)}<p\leq q} f\left(\frac{1}{pq\Psi(q)}\right)\\
&\leq \sum_{q=1}^{\infty} \sum_{1<p\leq q} f\left(\frac{1}{pq\Psi(q)}\right)\\
&\overset{(1.5)}{\asymp} \sum_{q=1}^{\infty} q f\left(\frac{1}{q^2\Psi(q)}\right)\\
&\leq \sum_{q=1}^{\infty} q\log(\Psi(q)) f\left(\frac{1}{q^2\Psi(q)}\right).
\end{aligned}$$

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Philip Bos, Department of Mathematical and Physical Sciences, La Trobe University, Bendigo 3552, Australia.

E-mail address: phil@philbos.com

Mumtaz Hussain, Department of Mathematical and Physical Sciences, La Trobe University, Bendigo 3552, Australia.

E-mail address: M.Hussain@latrobe.edu.au

David Simmons, The University of York, England, UK

E-mail address: David.Simmons@york.ac.uk